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Cascades and multifilters

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Abstract

Cascades (trees every element of which is a filter on the set of its successors), and multifilters, maps from cascades, are introduced. Multisequences constitute a special case of multifilters. Applications to convergence and to topology are indicated. © 2000 Elsevier Science B.V. All rights reserved.

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1. Introduction

Bifilters, the filters on sets of filters, and their contours, have proved their usefulness in topology; for example, Frolík applied contours of bifilters (that he called *sums of filters*) to a ZFC proof of the nonhomogeneity of the remainder of $\beta\mathbb{N}$ [8]. A slightly more general object is obtained from a filter $\mathcal{F}$ on Y and a map $m : Y \to \varphi(X)$. [2] The *contour* of $\mathcal{F}$ on m is defined by

$$\int_{\mathcal{F}} m = \bigcup_{F\in\mathcal{F}} \bigcap_{y\in F} m(y). \tag{1.1}$$

Such contours have been used by many authors (for instance, to characterize topologicity and regularity of convergences). Cook and Fischer call (1.1) the *compression operator* of $\mathcal{F}$ relative to m [2] and Kowalsky a *diagonal filter* [10]. Formula (1.1) has a meaning not only for filters $\mathcal{F}, m(y)$, but for arbitrary families of sets; such general contours were used in [9,3].

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[2] $\varphi(X)$ stands for the set of filters on X.

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In this short paper we introduce a notion of cascade of filters (or more generally of semifilters) and a related concept of multifilters of arbitrary ordinal rank (bifilters are multifilters of rank 2), and of their contours. A cascade of filters is a well-capped tree with root, every element of which is a filter on the set of its immediate successors. Multifilters are maps from cascades. A concept of decomposable prefilter of Aniskovič [1] amounts to that of the contour of a multisequence (roughly speaking, of a sequential cascade of countable rank).

Classically, the occurrence of filters in topology stems from the equivalence

$$x \in \mathrm{cl}_\tau A \Leftrightarrow \underset{\mathcal{F} \in \varphi A}{\exists} x \in \lim_\tau \mathcal{F}, \tag{1.2}$$

where $A \subset X$ and τ is a topology on X.[3] Let $\varphi^\infty A$ denote the class of all multifilters starting at A. A fundamental fact about multifilters is (Theorem 7.1) that, given a convergence ξ on X,

$$x \in \mathrm{cl}_\xi A \Leftrightarrow \underset{\Phi \in \varphi^\infty A}{\exists} x \in \lim_\xi \Phi,$$

where cl_ξ is the topological closure associated with the convergence ξ.

The method of multifilters is applied, in an essential way, to the problem of commutativity of the topologizer with products in the companion paper [5]. Multisequences are sequential multifilters, that is, maps from cascades all the elements of which are generated by sequences; they have been used to study the sequential order of product topologies [6, 11].

2. Cascades

If $(W, \sqsubseteq)$ is an ordered set, then we write

$$W(w) = \{x \in W \colon w \sqsubseteq x\}. \tag{2.1}$$

If W is well-founded,[4] then for every non maximal element w of W, there exists the set of immediate successors of w, that is,

$$W_+(w) = \min\{x \in W \colon w \sqsubset x\}.$$

Moreover there exists the unique *level*[5] function such that $l(w) = 0$, if $w \in \min W$, and otherwise

$$l(w) = l_W(w) = \sup_{v \sqsubset w} \bigl(l(v) + 1\bigr).$$

If W is a tree[6] then for each $u, w \in W$, the sets $W(u)$ and $W(w)$ are either disjoint or one includes the other.

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[3] A filter on a subset A of X converges to $x \in X$ for the convergence on X if its natural extension to X does.

[4] Every non empty subset contains a minimal element.

[5] Called also (lower) *rank*.

[6] $\{v \in W \colon v \sqsubset w\}$ is well-ordered for every $w \in W$. Each tree is well-founded.

An ordered set $(W, \sqsubseteq)$ is *well-capped* [7] if its every nonempty subset has a maximal point.[7] Each well-capped set admits the (upper) rank to the effect that $r(w) = 0$ if $w \in \max W$, and for $r(w) > 0$,

$$r(w) = r_W(w) = \sup_{v \sqsupset w} \bigl(r(v) + 1\bigr). \tag{2.2}$$

It follows from the definition that each element of a well-capped tree is of finite level (called also the *length*).

A well-capped tree with least element is called a *cascade*; the least element of a cascade V is denoted by $\emptyset = \emptyset_V$ and is called the *estuary* of V. In view of (2.1), $V(\emptyset_V) = V$. The *rank* of a cascade is by definition the rank of its estuary. A cascade V is said to be *on* A if $\max V \subset A$.

A cascade is a *filter cascade* if its every (non maximal) element is a filter on the set of its immediate successors. A *semifilter cascade* is a cascade every (non maximal) element of which is a semifilter[8] on the set of its immediate successors. In the sequel we will be primarily concerned with filter cascades, so that a "cascade" will mean a "filter cascade" if not specified differently. Maximal elements of a filter cascade can be identified with their principal ultrafilters.

By definition, two cascades of rank 0 are *equivalent*; two cascades V and W of nonzero rank are *equivalent* if there exist $F \in \emptyset_V$ and a bijective map $f : F \to W_+(\emptyset_W)$ such that $f(F) \in \emptyset_W$ and $V(v)$ is equivalent to $W(f(v))$ for every $v \in F$. Two equivalent cascades can be of different rank; however, as we will see later their limiting ranks are equal.

A cascade V is a *subcascade* of a cascade W if there exists an injective mapping $g : V \to W$ such that for every non maximal $v \in V$,

$$g(\emptyset_V) = \emptyset_W, \tag{2.3}$$

$$g\bigl(V_+(v)\bigr) \subset W_+\bigl(g(v)\bigr), \tag{2.4}$$

$$g(v) \leqslant g_\natural(v), \tag{2.5}$$

where $g(v)$ is the image of v as an element of W, and $g_\natural(v)$ is the filter generated by $\{g(F){:}\ F \in v\}$. It follows that $l(v) = l(g(v))$.

Two families of sets $\mathcal{A}$ and $\mathcal{B}$ are said to *mesh*, in symbols, $\mathcal{A}\#\mathcal{B}$, if $A \cap B \neq \emptyset$ for every $A \in \mathcal{A}$ and $B \in \mathcal{B}$. If $\mathcal{B} = \{B\}$, then we abridge $B\#\mathcal{A}$ for $\mathcal{B}\#\mathcal{A}$. The *grill* $\mathcal{A}^\#$ of $\mathcal{A}$ is $\{H{:}\ H\#\mathcal{A}\}$.

A subset X of a cascade V is *frequent* if $\emptyset_V \in X$, and $X_+(v) \subset V_+(v)$ and $X_+(v) \in v^\#$[9] for each non maximal $v \in X$. Conditions (2.4) and (2.5) imply that if $g : V \to W$ defines a subcascade, then $g(V)$ is frequent in W.

A subcascade V of W is a *frequent subcascade* if (2.5) is strengthened to become

$$g_\natural(v) = g(v) \vee g\bigl(V_+(v)\bigr). \tag{2.6}$$

[7] In other words, a well-capped ordered set is a well-founded ordered set for the inverse order.
[8] A *semifilter* $\mathcal{F}$ on A is a family of subsets of A such that $F \in \mathcal{F}$ and $F \subset G$ imply $G \in \mathcal{F}$.
[9] X is well-capped as a subset of a well-capped set.

A cascade L on a set of cascades $\mathfrak{M}$ can be extended in a natural way to a cascade $L \hookleftarrow \mathfrak{M}$ called the *confluence* (of $\mathfrak{M}$ to L):

$$L \setminus \max L \cup \bigcup_{M \in \mathfrak{M}} M,$$

where the extension of the existent orders is defined by

$$v \in L,\ x \in M \in \mathfrak{M}, \quad v \sqsubseteq_L M \Rightarrow v \sqsubseteq x.$$

Each cascade L is the confluence of the cascades $L(w)$ with $l(w) = 1$ to the estuary: $L = \emptyset_L \hookleftarrow \{L(w)\colon\ w \in L_+(\emptyset_L)\}$. If $\mathfrak{M}$ is a set of cascades on a set of cascades $\mathfrak{P}$, then we adopt the convention that

$$\mathfrak{M} \hookleftarrow \mathfrak{P} = \{M \hookleftarrow \mathfrak{P}\colon\ M \in \mathfrak{M}\}.$$

The following associative law holds:

Proposition 2.1. *If L is a cascade on $\mathfrak{M}$ and each $M \in \mathfrak{M}$ is a cascade on a set $\mathfrak{P}$ of cascades on X, then*

$$(L \hookleftarrow \mathfrak{M}) \hookleftarrow \mathfrak{P} = L \hookleftarrow (\mathfrak{M} \hookleftarrow \mathfrak{P}). \tag{2.7}$$

3. Limiting rank

What really counts for cascades from the point of view of convergence theory is not so much their rank, but their limiting rank. Define the *limiting rank* $\varrho(v) = \varrho(v; V)$ of an element v of a cascade V by well-founded induction: if $v \in \max V$, then $\varrho(v) = 0$; if $v \notin \max V$ and ϱ has been defined for every $w \sqsupset v$, then

$$\varrho(v) = \sup_{F \in v} \inf_{w \in F} \big(\varrho(w) + 1\big). \tag{3.1}$$

By definition, the *rank* $r(V)$ (respectively, *limiting rank* $\varrho(V)$) of a cascade V is the rank of its estuary. A cascade V is *monotone* if $r_V(v) = \varrho_V(v)$ for every $v \in V$.

Theorem 3.1. *Every cascade V admits a frequent monotone subcascade W such that $\varrho(V) = r(W)$.*

Proof. We induce on $\varrho(v)$. The claim is obviously true for $\varrho(v) = 0$. Suppose that the theorem is proved for all w with $\varrho(w) < \alpha$ and let $\alpha = \varrho(v)$. Then for every $w \in V_+(v)$, there exists a monotone frequent subcascade $g_w : W_w \to V(w)$ such that $\varrho(W_w) = \varrho_V(w)$. It follows from a general rule concerning grills [9] that

$$\alpha = \sup_{F \in v} \inf_{w \in F} \big(\varrho_V(w) + 1\big) = \min_{H \in v^{\#}} \sup_{w \in H} \big(\varrho_V(w) + 1\big),$$

there exists $H \in v^{\#}$ such that $\alpha = \sup_{w \in H}(\varrho_V(w) + 1)$. Let $w_0 = v \vee H$, and $W = \{(w_0, x)\colon\ x \in W_w, w \in H\}$; set $g(w_0) = \emptyset_V$ and $g(w_0, x) = g_w(x)$ for $x \in W_w$. The constructed W is a required subcascade. □

4. Contours of cascades

Let V be a cascade. The *contour* $\int V$ of V is defined by induction on the rank:

$$r(V) = 1 \Rightarrow \int V = \emptyset_V; \tag{4.1}$$

$$r(V) > 1 \Rightarrow \int V = \bigcup_{F \in \emptyset_V} \bigcap_{w \in F} \int V(w). \tag{4.2}$$

Given a family $\mathfrak{M}$ of cascades, we adopt the convention that

$$\int \mathfrak{M} = \left\{ \int M\colon\ M \in \mathfrak{M} \right\}.$$

Proposition 4.1. *The contour of a confluence of $\mathfrak{M}$ to L is equal to the contour of the confluence of $\int L$ to $\int \mathfrak{M}$:*

$$\int (L \hookleftarrow \mathfrak{M}) = \int \left(\int L \hookleftarrow \int \mathfrak{M} \right). \tag{4.3}$$

Proof. If $r(L) = 1$, then (4.3) amounts to the definition of the contour (in this case $\int L = \emptyset_L$). Suppose that (4.3) holds for each cascade L of rank less than α, and that $r(V) = \alpha$. Let $\mathfrak{L} = \{V(w)\colon\ w \in V_+(\emptyset_V)\}$. By (2.7), $V \hookleftarrow \mathfrak{M} = (\emptyset_V \hookleftarrow \mathfrak{L}) \hookleftarrow \mathfrak{M} = \emptyset_V \hookleftarrow (\mathfrak{L} \hookleftarrow \mathfrak{M})$. Therefore, as $r(\emptyset_V) = 1$, by (4.3) and by inductive hypothesis,

$$\int (V \hookleftarrow \mathfrak{M}) = \int \left(\emptyset_V \hookleftarrow \int (\mathfrak{L} \hookleftarrow \mathfrak{M}) \right).$$

As $r(V(w)) < \alpha$ for every $V(w) \in \mathfrak{L}$, by inductive hypothesis,

$$\begin{aligned}
\int (V \hookleftarrow \mathfrak{M}) &= \int \left(\emptyset_V \hookleftarrow \int \left(\int \mathfrak{L} \hookleftarrow \int \mathfrak{M} \right) \right) \\
&= \int \left(\left(\emptyset_V \hookleftarrow \int \mathfrak{L} \right) \hookleftarrow \int \mathfrak{M} \right) \\
&= \int \left(\int V \hookleftarrow \int \mathfrak{M} \right). \qquad \square
\end{aligned}$$

A cascade L is said to *mesh* a set A if $A \in (\int L)^{\#}$. With each cascade of semifilters L we can associate the cascade of semifilters $L^{\#}$ defined with aid of the following bijective map $f : L \to L^{\#}$: if $r(v) = 0$, then $f(v) = v$; if $r(v) > 0$, and $f(w)$ has been defined for every $w \in L$ with $r(w) < r(v)$, then $f(v) = (f_{\natural}(v))^{\#}$. We consider $L^{\#}$ with the order induce by the bijection f.

Theorem 4.2. *For each cascade V,*

$$\left(\int V \right)^{\#} = \int V^{\#}. \tag{4.4}$$

Proof. The claim is obviously true for $r(V)=0,1$. If $r(V)=2$ then (4.4) amounts to [9]

$$\left(\bigcup_{F\in\emptyset}\bigcap_{w\in F} w\right)^{\#}=\bigcap_{H\in\emptyset^{\#}}\bigcup_{w\in H} w^{\#}. \tag{4.5}$$

Assume the claim is true for every cascade of rank less than $\alpha>2$, and let V be a cascade of rank α. Consider $\mathcal{V}=\{V(v)\colon v\in V_{+}(\emptyset)\}$. Then by (4.5),

$$\left(\int V\right)^{\#}=\left(\int(\emptyset_V \looparrowleft \mathcal{V})\right)^{\#}=\left(\int\left(\emptyset_V \looparrowleft \int\mathcal{V}\right)\right)^{\#}=\int\left(\emptyset_V^{\#} \looparrowleft \left(\int\mathcal{V}\right)^{\#}\right).$$

By inductive hypothesis, $(\int\mathcal{V})^{\#}=\int\mathcal{V}^{\#}$, so that $(\int V)^{\#}=\int(\emptyset_V^{\#}\looparrowleft\int\mathcal{V}^{\#})=\int V^{\#}$. □

By definition,

$$\left(\int(L\looparrowleft\mathfrak{M})\right)^{\#}=\int(L^{\#}\looparrowleft\mathfrak{M}^{\#}),$$

for each cascade L on a family $\mathfrak{M}$ of cascades. Consequently, if $r(L)=1$, then $A\#(\int L)$ whenever $A\#\emptyset_L$. If $r(L)>1$, $A\#(\int L)$ if there exists $H\in\emptyset_L^{\#}$ such that $A\#(\int L(w))$ for each $w\in H\cap L_{+}(\emptyset_L)$.

Corollary 4.3. *A cascade L admits a frequent subcascade on A if and only if $(\int L)\#A$.*

5. Multifilters

A map $\Phi: V\backslash\{\emptyset_V\}\to X$, where V is a cascade, is called a *multifilter on X*. In order not to overburden the notation, we will talk about a multifilter $\Phi: V\to X$, under the understanding that Φ is not defined at $\emptyset_V$. If $\Phi(\max V)\subset A\subset X$, then we say that Φ *starts at A*; the class of multifilters starting at A is denoted by $\varphi^{\infty}(A)$.

The notion of multifilter that starts at A is a natural extension of that of filter on A. [10] Actually what really extends the notion of filter is the map

$$\Phi_{\natural}: V\setminus\max V\to\varphi(X), \tag{5.1}$$

associated with Φ, where $\Phi_{\natural}(v)$ is the filter generated by $\{\Phi(F)\colon F\in v\}$. (However a systematic distinction of the two concepts would complicate the presentation.) If Φ is a multifilter of rank 1, then $\Phi_{\natural}:\{\emptyset_V\}\to\varphi(X)$ is the a map from a singleton consisting of the filter $\emptyset_V$ on $\max V$, so that $\Phi_{\natural}(\emptyset_V)$ is a filter on $\Phi(\max V)$.

A couple (V,Φ_0) where V is a cascade and $\Phi_0:\max V\to A$ is called a *perifilter on A*. In the sequel we will consider V implicitly talking about a perifilter Φ_0. If $\Phi|_{\max V}=\Phi_0$, then we say that the multifilter Φ is an *extension* of the perifilter Φ_0. The rank (limiting rank) of a multifilter (perifilter) is, by definition, the rank (limiting rank) of the corresponding cascade. A perifilter and a multifilter from a monotone cascade are said to be *monotone*.

[10] In fact a filter defined on X that contains A.

A multifilter is called *free*, *sequential*, *ultra-*, and so on, if all its values are respectively free filters, sequences, ultrafilters, and so on. A *submultifilter* of a multifilter $\Phi : W \to X$ is a multifilter $\Psi : V \to X$ such that V is a subcascade of W, and the defining function $g : V \to W$ from (2.3)–(2.5) fulfills $\Psi = \Phi \circ g$.

A multifilter $\Phi : V \to X$ is called *simple* if for every $v \in V$ and $w, y \in V_+(v)$ such that $\Phi(w) = \Phi(y)$, the filters $\Phi_\natural(w)$ and $\Phi_\natural(y)$ coincide. A submultifilter of a simple multifilter is simple. Multifans and arrows [6] are examples of multifilters that are not simple.

If $\Phi : V \to X$ is a multifilter, and if for each $v \in \max V$, a multifilter $\Psi_v : W_v \to X$ is such that $\Phi(v) = \Psi_v(\emptyset)$, then on setting $\mathfrak{V} = \{\Psi_v\colon\ v \in \max V\}$,

$$\Phi \hookleftarrow \mathfrak{V}$$

is the confluence of $\mathfrak{V}$ to Φ, that is, the multifilter from $V \hookleftarrow \{W_v\colon\ v \in \max V\}$ to X such that $(\Phi \hookleftarrow \mathfrak{V})(v) = \Phi(v)$ if $v \in V$, and $(\Phi \hookleftarrow \mathfrak{V})(w) = \Psi_v(w)$ if $w \in W_v$.

Example 5.1. A *multisequence* on X [6] is an application from $\max T$ to X, where T is a tree of multi-indices, that is, a well-capped subset of $\mathrm{Seq} = \bigcup_n \mathbb{N}^n$, the set of all finite sequences of natural numbers (ordered by inclusion $\sqsubseteq$), verifying

$$\forall s, t \in \mathrm{Seq}\ t \in T \text{ and } s \sqsubseteq t \Rightarrow s \in T, \tag{5.2}$$

$$\exists n \in \mathbb{N}\ (t, n) \in T \Rightarrow \forall n \in \mathbb{N}\ (t, n) \in T, \tag{5.3}$$

where $(s, t) = (n_0, \ldots, n_k, m_0, \ldots, m_p)$ if $s = (n_0, \ldots, n_k)$ and $t = (m_0, \ldots, m_p)$. A multisequence is called *monotone* if the rank $r(t, n)$ (with respect to the corresponding tree T of multi-indices) is increasing for every $t \in T$. An *extended multisequence* on X is a map from a tree of multi-indices to X. Each tree of multi-indices T can be identified with a free sequential cascade: if $t \in T \setminus \max T$, then t is the cofinite filter on $\{(t, n)\colon\ n \in \omega\}$, equivalently, is generated by the sequence $((t, n))_n$. A multisequence is a sequential perifilter, and an extended multisequence is a sequential multifilter. A sequence is a multisequence of rank 1; a multisequence of rank 2 is called a *bisequence*, and so on.

Fremlin used in [7] *sequentially regular embeddings*, that is, injective convergent multisequences without convergent transverse sequences. Aniskovič [1] defines *decomposable prefilters* that are the contours of free sequential cascades.

6. Contours of multifilters

The contour of a multifilter $\Phi : V \to X$ depends entirely on the underlying cascade V and on the restriction of Φ to $\max V$, hence on the corresponding perifilter $(V, \Phi|_{\max V})$. Therefore we shall not distinguish between the contours of multifilters and of the corresponding perifilters. The *contour* of $\Phi : W \to X$ is defined by induction to the effect that $\int \Phi = \Phi_\natural(\emptyset_W)$ if $r(\Phi) = 0$, and

$$\int \Phi = \bigcup_{F \in \emptyset_W} \bigcap_{w \in F} \int \Phi|_{W(w)} \tag{6.1}$$

otherwise. In particular, if Φ is a multifilter of rank 1, then $\int \Phi = \Phi_\natural(\emptyset)$; if Φ is a multifilter of rank 2, then (6.1) becomes

$$\int \Phi = \bigcup_{F \in \emptyset_W} \bigcap_{w \in F} \Phi_\natural(w).$$

For the “identity multifilter” $I : W \to W$, the contour coincides with the contour of the underlying cascade. More generally,

Proposition 6.1. *For each multifilter $\Phi : W \to X$,*

$$\Phi\left(\int W\right) = \int \Phi.$$

Proof. Indeed, $B \in \Phi(\int W)$ if and only if there exists $A \in \int W$ with $B \supset \Phi(A)$; equivalently, there exists $F \in \emptyset_W$ such that $A \in \int W(w)$ for every $w \in F$, and $B \supset \Phi(A)$; in other words, $B \in \Phi(\int W(w))$ for each $w \in F$. By inductive assumption, $B \in \int \Phi|_{W(w)}$ for each $w \in F$, hence $B \in \int \Phi$. □

We say that a multifilter $\Phi : W \to X$ *meshes* a set A if $(\int \Phi) \# A$.

Example 6.2 (Compression operator, diagonal filter). If $\mathcal{F}$ is a filter on Y [11] and $m : Y \to \varphi(X)$, then the *contour* of $\mathcal{F}$ on m is defined by

$$\int_{\mathcal{F}} m = \bigcup_{F \in \mathcal{F}} \bigcap_{y \in F} m(y). \tag{6.2}$$

If $\mathcal{F}$ is a filter on a set $\mathfrak{M}$ of filters, and $i_{\mathfrak{M}}$ stands for the identity on $\mathfrak{M}$, then we abridge the notation $\int \mathcal{F} = \int_{\mathcal{F}} \mathfrak{M} = \int_{\mathcal{F}} i_{\mathfrak{M}}$.

Eq. (6.2) is a special case of contour of multifilter of rank 2. In fact, define a cascade by $\emptyset = \mathcal{F}$ and identify each $y \in Y$ with the filter $m(y)$ defined on a copy X_y of X. Then $\Phi(y, x) = x$ for every $(y, x) \in \max V$ defines a perifilter for which $\Phi_\natural(y) = m(y)$ and thus $\int_{\mathcal{F}} m = \int \Phi$.

Vice versa, if $\Phi : V \to X$ is a multifilter of rank 2, then $\emptyset$ is a filter on $V_+(\emptyset)$, and $m(v) = \Phi_\natural(v)$ is a filter on X for every $v \in V_+(\emptyset)$. Therefore by (6.2), $\int \Phi = \int_\emptyset \Phi_\natural$. Writing in extenso,

$$\int \Phi = \bigcup_{F \in \emptyset} \bigcap_{v \in F} \Phi_\natural(v) = \bigcup_{G \in \Phi_\natural(\emptyset)} \bigcap_{x \in G} \left(\bigcap_{v \in \Phi^-(x)} \Phi_\natural(v) \right).$$

Example 6.3 (Case of simple multifilters). If $\Phi : V \to X$ is a simple bifilter, then $\mathcal{F} = \Phi_\natural(\emptyset)$ is a filter on $Y = \Phi(V_+(\emptyset))$, and for every $x \in \Phi(V_+(\emptyset))$ the filter $\Phi_\natural(v)$ is independent of the choice of $v \in \Phi^-(x)$. Therefore,

$$m(x) = \Phi_\natural\big(\Phi^-(x)\big)$$

[11] The same definition makes sense for semifilters [3].

is a map from $Y = \Phi(V_+(\emptyset))$ to $\varphi(X)$. Then

$$\int_{\mathcal{F}} m = \bigcup_{F \in \Phi_\natural(\emptyset)} \bigcap_{x \in F} \Phi_\natural\bigl(\Phi^-(x)\bigr) = \bigcup_{H \in \emptyset} \bigcap_{v \in H} \Phi_\natural(v) = \int \Phi.$$

7. Convergence

A *convergence* ξ on a set X is a relation between X and the set of filters on X, denoted $x \in \lim_\xi \mathcal{F}$, if x and $\mathcal{F}$ are ξ-related, for which $\lim_\xi \mathcal{F} \subset \lim_\xi \mathcal{G}$ whenever $\mathcal{F} \subset \mathcal{G}$, and such that $x \in \lim_\xi (x)$, where (x) stands for the principal filter of x, for each $x \in X$. A subset A of X is ξ*-closed* if $A \in \mathcal{F}$ implies that $\lim_\xi \mathcal{F} \subset A$. The family of ξ-closed sets generates a topology denoted by $T\xi$ and called the *topologization* of ξ. We denote by $\mathrm{cl}_\xi A$ the topological closure of A, that is, the least ξ-closed set that includes A. A map $f:(X,\xi) \to (Y,\tau)$ between convergence spaces is continuous if $f(\lim_\xi \mathcal{F}) \subset \lim_\tau f(\mathcal{F})$ for each filter $\mathcal{F}$ on X.

Let ξ be a convergence on X. A multifilter $\Phi : V \to X$ *converges* to $x_0 \in X$ (in symbols, $x_0 \in \lim_\xi \Phi$) if

$$\forall_{v \in V \setminus \max V \setminus \{\emptyset_V\}} \quad \Phi(v) \in \lim_\xi \Phi_\natural(v), \tag{7.1}$$

$$x_0 \in \lim_\xi \Phi_\natural(\emptyset_V), \tag{7.2}$$

where $\Phi(v)$ is the image by Φ of v treated as a point of V, while $\Phi_\natural(v)$ is the filter generated by $\{\Phi(F)\colon\ F \in v\}$. It follows that if a multifilter converges to x, then its every submultifilter converges to x.

By definition of *adherence*,

$$x \in \mathrm{adh}_\xi A \Leftrightarrow \exists_{\mathcal{F} \in \varphi A} \ x \in \lim_\xi \mathcal{F}. \tag{7.3}$$

For each subset A, let $\mathrm{adh}^0_\xi A = A$, and for each ordinal $\alpha > 0$, let

$$\mathrm{adh}^\alpha_\xi A = \mathrm{adh}_\xi \Bigl(\bigcup_{\beta<\alpha} \mathrm{adh}^\beta_\xi A\Bigr) \quad \text{and} \quad \mathrm{adh}^{<\alpha}_\xi A = \bigcup_{\beta<\alpha} \mathrm{adh}^\beta_\xi A.$$ [12]

For every α, the iterated adherence $\mathrm{adh}^\alpha_\xi A$ is included in $\mathrm{cl}_\xi A$. The least α such that $\mathrm{adh}^\alpha_\xi A = \mathrm{cl}_\xi A$ for every $A \subset X$ is called the *topological defect* of ξ and is denoted by $t(\xi)$. The reason for introducing cascades and multifilters is the following

Theorem 7.1. *Let ξ be a convergence. Then*

$$x \in \mathrm{adh}^\alpha_\xi A \tag{7.4}$$

if and only if there exists a (*monotone, simple*) *multifilter of rank less than or equal to α that starts from A and ξ-converges to x.*

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[12] Given a family $\mathfrak{F}$ of filters, we denote by $\lim_\xi \mathfrak{F}$ the union of $\lim_\xi \mathcal{F}$ over $\mathcal{F} \in \mathfrak{F}$. Consequently, $\mathrm{adh}_\xi A = \lim_\xi \varphi A$. As a result iterations of an adherence constitute a sequel of alternating operations: φ, associating with each set, the set of its filters, and $\lim_\xi$, associating with each set of filters, the union of their limits.

Proof. Fix a convergence on X and let $A \subset X$. First we show that $\lim \Phi \subset \operatorname{adh}^{\alpha} A$ for each multifilter Φ starting from A with $r(\Phi) < \alpha$. For $\alpha = 0$, this follows from the definition. If $\Phi : \max V \to A$ is a multifilter that converges to x, then $\Phi_{\natural}(\emptyset)$ is a filter on A that converges to x. Hence by (7.3) $\lim \Phi \subset \operatorname{adh} A$. If $\Phi : V \to X$ is a multifilter of rank $\alpha > 1$ that starts from A and converges to x, then for each $v \in V_{+}(\emptyset)$, the restriction of Φ to $V(v)$ starts from A and, by inductive hypothesis, converges to an element of $\operatorname{adh}^{<\alpha} A$. Thus by the first step, $x \in \operatorname{adh}^{\alpha} A$.

If (7.4) holds for $\alpha = 0$, then $x \in A$, and the principal filter of x defines a multifilter of rank 0 that converges to x. Suppose that the condition is fulfilled for all $\alpha < \gamma$ and let $x \in \operatorname{adh}^{\gamma} A$. Then, by definition, there exists a filter $\mathcal{F}$ on $\operatorname{adh}^{<\gamma} A$ that converges to x. By inductive assumption for every $y \in \operatorname{adh}^{<\gamma} A$ there exist $\alpha(y) < \gamma$, a cascade V_y, and a simple multifilter $\Phi_y : V_y \to X$ of rank less than or equal to $\alpha(y)$ that starts from A and converges to y. Consider $\{\mathcal{F}\} \cup \operatorname{adh}^{<\gamma} A$ as a cascade with the estuary $\mathcal{F}$ and the maximal elements $\operatorname{adh}^{<\gamma} A$. Then

$$W = \left(\{\mathcal{F}\} \cup \operatorname{adh}^{<\gamma} A\right) \hookleftarrow \left\{V_y \colon\ y \in \operatorname{adh}^{<\gamma} A\right\}$$

is a cascade of rank less than or equal to γ. The map defined by $\Phi(v) = \Phi_y(v)$ if $v \in V_y$ is a simple multifilter that starts from A and converges to x. By Theorem 3.1 there exists a monotone (simple) submultifilter of Φ. □

Corollary 7.2. *Let ξ be a convergence. Then $x \in \operatorname{cl}_{\xi} A$ if and only if there exists a* (*monotone, simple*) *multifilter that starts from A and ξ-converges to x.*

Theorem 7.3. *Let ξ be a T_1-convergence and $\alpha > 0$.*[13] *If*

$$x \in \operatorname{adh}_{\xi}^{\alpha} A \setminus \operatorname{adh}_{\xi}^{<\alpha} A, \tag{7.5}$$

then there exists a (*monotone*) *free multifilter of rank α that starts from A and ξ-converges to x.*

Proof. If $\alpha = 1$, then (7.5) amounts to (7.4) and $x \notin A$. By definition, this implies the existence of a filter $\mathcal{F}$ on A that converges to x; this filter is free, for otherwise a principal ultrafilter of an element of A would converge to x, what is impossible by T_1.

Suppose that the claim is true for all $\alpha < \gamma$. By the implication just established for the rank 1, if $x \in \operatorname{adh}^{\gamma} A \setminus \operatorname{adh}^{<\gamma} A$, then there exists a free filter $\mathcal{F}$ on $\operatorname{adh}^{<\gamma} A$ that converges to x. By inductive assumption, if $w \in \operatorname{adh}^{<\gamma} A$, then there exists $\alpha(w) < \gamma$ and a free multifilter Φ_w starting from A of rank $\alpha(w)$ that converges to w. The multifilter Φ constructed in the proof of Theorem 7.1 is free and converges to x. The limiting rank $\varrho(\Phi)$ is not greater than γ, for otherwise by the inductive assumption $x \in \operatorname{adh}^{\varrho(\Phi)} A$, contrary to $x \notin \operatorname{adh}^{<\gamma} A$. By Theorem 3.1 there exists a monotone submultifilter of Φ. □

Simple examples show that in Theorem 7.3 it is not possible to use identity multifilters (that is, it is not possible to consider cascades directly on the convergence space) even in

[13] Every singleton is closed.

the case of (topologically) Hausdorff convergence of topological defect 2. Indeed consider the classical convergence defined on $X = \{\infty\} \cup \mathcal{A} \cup Y$ with the aid of maximal almost disjoint family $\mathcal{A}$ on an (infinite) set Y, to the effect that the points of Y are isolated, if $A \in \mathcal{A}$, then $A = \lim \mathcal{F}$ if $\mathcal{F}$ is finer than the cofinite filter on A, and $\infty = \lim \mathcal{F}$ if $\mathcal{F}$ is finer than the cofinite filter on $\mathcal{A}$. Then no cascade (i.e., identity multifilter) that starts from Y, converges to ∞, because otherwise there would exist maximal elements of such a cascade with several (actually infinitely many) immediate predecessors, contradicting the tree assumption.

Even if we relax the assumptions on order in the definition of cascade, we will be still forced, in the situation of Theorem 7.3, to use some analogues of non-identity multifilters. For example, define the following convergence on the disjoint union $X = \{a\} \cup B \cup \{\infty\}$ (with infinite B) for which, besides the principal ultrafilters converging to their defining points, B is the limit of the principal ultrafilter of a, and $\{\infty\} = \lim \mathcal{F}$ provided $\mathcal{F}$ is finer than the cofinite filter on B. A cascade-like object starting at a would be a principal-ultrafilter cascade and thus must not converge to ∞.

If $\mathfrak{J}$ is a class of filters, the elements of $\mathfrak{J}$ are called $\mathfrak{J}$-*filters*. A $\mathfrak{J}$-*multifilter* is a multifilter Φ for which $\Phi_\natural(v)$ is a $\mathfrak{J}$-filter for each non maximal element v. For example, multisequences are the $\mathfrak{J}$-multifilters, where $\mathfrak{J}$ is the family of all sequential filters.

Example 7.4. If ξ is a convergence, then the *sequential modification* $\operatorname{Seq}\xi$ of ξ is

$$x \in \lim_{\operatorname{Seq}\xi} \mathcal{F} \Leftrightarrow \underset{(x_n)}{\exists}\ x \in \lim_\xi (x_n) \quad \text{and} \quad \mathcal{F} \geqslant (x_n).$$

It is known that a topology ξ is sequential if and only if $\xi = T\operatorname{Seq}\xi$. The *sequential order* $\sigma(\xi)$ of a topology ξ is the topological defect of its sequential modification: $\sigma(\xi) = t(\operatorname{Seq}\xi)$. Thus Theorem 7.3 implies the following fact (see [6, Theorem 1.3]): if τ is a sequential T_1-topology of sequential order α, then $x \in \operatorname{cl}_\tau A$ if and only if there exists on A a monotone free multisequence of rank α that τ-converges to x.

A convergence ξ on X is said to be *diagonal* if for every filter $\mathcal{F}$ and for each map $m : X \to \varphi(X)$ such that $x \in \lim_\xi m(x)$,

$$\lim_\xi \mathcal{F} \subset \lim_\xi \int_{\mathcal{F}} m.$$

It follows from Example 6.3 with the aid of induction on the rank that if a convergence is diagonal, then for every simple multifilter Φ,

$$\lim \Phi \subset \lim \int \Phi. \tag{7.6}$$

The *diagonalization* $D\xi$ of ξ is the finest among the diagonal convergences that are coarser than ξ. It is straightforward that

Proposition 7.5. *An element $x \in \lim_{D\xi} \mathcal{F}$ if and only if there exists a simple multifilter Φ such that $x \in \lim_\xi \Phi$ and $\mathcal{F} \geqslant \int \Phi$.*

Of course, each topology is diagonal. In view of Example 6.2, [4, Corollary 8.4] can be reworded to the effect that a convergence is a topology if and only if (7.6) holds for each (not only simple!) bifilter (multifilter) Φ. Moreover,

Proposition 7.6. *An element $x \in \lim_{T\xi} \mathcal{F}$ if and only if there exists a multifilter Φ such that $x \in \lim_{\xi} \Phi$ and $\mathcal{F} \geqslant \int \Phi$.*

8. Compactness

Let ξ be a convergence on X and let $\mathfrak{J}$ be a class of filters. A subset K of X is ξ-$\mathfrak{J}$-*compact* (shortly, $\mathfrak{J}$-*compact*) if $\operatorname{adh}_{\xi} \mathcal{H} \cap K \neq \emptyset$ for each $\mathfrak{J}$-filter $\mathcal{H}$ such that $\mathcal{H}\#K$. For example, $\mathfrak{J}$-compactness in the case of $\mathfrak{J}$, the class of all filters, is the classical compactness, and for $\mathfrak{J}$, the set of all countably based filters, $\mathfrak{J}$-compactness amounts to countable compactness. A class $\mathfrak{J}$ of filters is said to be *composable* if it contains all principal filters, and if $\mathcal{H}\mathcal{F} = \{HF\colon\ H \in \mathcal{H},\ F \in \mathcal{F}\} \in \mathfrak{J}$ for every $\mathcal{F}, \mathcal{H} \in \mathfrak{J}$.

Theorem 8.1. *Let $\mathfrak{J}$ be a composable class of filters. Every $\mathfrak{J}$-perifilter that meshes a $\mathfrak{J}$-compact set K has a subperifilter that can be extended to a multifilter starting from K that converges to an element of K.*

Proof. We carry out the proof by induction on the rank of the perifilter. This is obvious for the rank 0 or 1. Assume that the claim is true for the rank less than α and let $\Lambda : \max V \to X$ be a $\mathfrak{J}$-perifilter of rank α that meshes K. Therefore, for every $v \in V_+(\emptyset)$, the restriction Λ_v of Λ to $\max V(v)$ is a $\mathfrak{J}$-perifilter of rank less than α that meshes K. By inductive assumption, for every $v \in V_+(\emptyset)$, there exists a subcascade $g_v : W_v \to V(v)$, and a multifilter $\Phi_v : W_v \to X$ that is an extension of $g_v \circ \Lambda_v$, and an element x_v of K such that $x_v \in \lim_{\xi} \Phi_v$. By composability, the filter (generated by) $\{\{x_v\colon\ v \in F\}\colon\ F \in \emptyset_V\}$ belongs to $\mathfrak{J}$ because $\emptyset_V$ is a $\mathfrak{J}$-filter, and contains K. Hence there exists an ultrafilter $\mathcal{U}$ finer than $\{\{x_v\colon\ v \in F\}\colon\ F \in \emptyset_V\}$ that converges to an element x of K. Let $W = \{x\} \cup \bigcup_{v \in V_+(\emptyset)} W_v$ be a cascade and define $\Phi : W \to X$ by $\Phi(v, w) = \Phi_v(w)$ for $w \in W_v$. Then Φ is an extension of a subperifilter of Λ that converges to an element of K. $\square$

By Theorem 8.1 and Proposition 7.5, in countably compact topological spaces, the contours of the multifilters of countably based filters have non empty adherence. In general such contours are not countably based. If $\mathfrak{J}$ is a class of filters, let $\operatorname{contour}(\mathfrak{J})$ denote the class of contours of $\mathfrak{J}$-multifilters. Then $\operatorname{contour}(\mathfrak{J})$-compactness arises as a natural extension to convergences of the topological notion of $\mathfrak{J}$-compactness.

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